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The fundamental solution and Fourier series in eigenfunctions of the magnetic Schrödinger operator

V S Serov

Department of Mathematical Sciences, University of Oulu, PO Box 3000, FIN-90014, Oulu, Finland

E-mail: vserov@cc.oulu.fi

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Abstract
We study the magnetic Schrödinger differential operator in the arbitrary domains. The coefficients of this operator are assumed to be in L^s and W_s^1 spaces, respectively to the derivatives of order 0 and order 1. We prove the existence of the fundamental solution (as well as the existence of the Green's function for these domains) and its uniform estimates. We obtain the conditions which guarantee the absolute and uniform convergence of the Fourier series in eigenfunctions up to the boundary of a bounded domain. These results might be applied to the ground of the Fourier method.

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1. Introduction

Let Ω be an arbitrary domain with a smooth boundary in $R^n, n \geqslant 3$. We consider in this domain a magnetic Schrödinger operator

$$H_m = -(\nabla + \mathrm{i}\vec{A}(x))^2 + V(x)\cdot, \qquad x \in \Omega, \tag{1}$$

where the coefficients $\vec{A}(x)$ (magnetic potential) and $V(x)$ (electric potential) are assumed to be real-valued. We assume also that $\vec{A}(x) \in \left(W_s^1(\Omega)\right)^n$ and $V(x) \in L^s(\Omega)$ for some $\frac{n}{2} < s \leqslant \infty$.

It is well known (see [18–10]) that under the above-mentioned conditions for the coefficients, symmetric operator (1) can be defined by the method of quadratic forms. It can be proved that for any function $u \in C_0^\infty(\Omega)$ the following Gårding's inequality holds:

$$(H_m u, u)_{L^2} \geqslant c_1 \|\nabla u\|_{L^2}^2 - c_2 \|u\|_{L^2}^2, \tag{2}$$

where $0 < c_1 < 1, c_2 > 0$.

Since H_m is symmetric and semi-bounded from below H_m has a self-adjoint Friedrichs extension denoted by $(H_m)_F$ in $L^2(\Omega)$ with domain

$$D((H_m)_F) = \left\{ f(x) \in \overset{\circ}{W}{}_2^1(\Omega) : H_m f(x) \in L^2(\Omega) \right\}, \tag{3}$$

where $\overset{\circ}{W}{}_2^1(\Omega)$ denotes the closure of the space $C_0^\infty(\Omega)$ by the norm of Sobolev space $W_2^1(\Omega)$. If Ω is bounded then the spectrum of this extension is purely discrete, finite multiplicity and has an accumulation point only at $+\infty$:

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots \to +\infty.$$

The corresponding orthonormal eigenfunctions $\{\varphi_k(x)\}_{k=1}^\infty$ form an orthonormal basis in $L^2(\Omega)$.

We study the absolute and uniform convergence of the Fourier series associated with the operator $(H_m)_F$ for various classes of differentiable functions.

In our case to each function $f \in L^2(\Omega)$, we can assign the formal series

$$f \sim \sum_{k=1}^\infty f_k \varphi_k(x), \tag{4}$$

where f_k are the Fourier coefficients of f with respect to the system $\{\varphi_k(x)\}_{k=1}^\infty$ of eigenfunctions. In this work, we will establish the following theorems.

Theorem 1. *Suppose that $\vec{A}(x) \in \left(W_s^1(\Omega)\right)^n$ and $V(x) \in L^s(\Omega)$ for some $\frac{n}{2} < s \leqslant \infty$. Then there exist $C > 0$ and $\lambda_0 > 0$ such that for any $\lambda \geqslant \lambda_0$, the operator $H_m + \lambda I$ has a fundamental solution $F(x, y, \lambda)$ which satisfies the following estimate:*

$$|F(x, y, \lambda)| \leqslant C|x - y|^{2-n}\, \mathrm{e}^{-\frac{1}{2}|x-y|\sqrt{\lambda}}, \tag{5}$$

for all $x, y \in \Omega$.

Without loss of generality, in the following theorem we assume that $(H_m)_F$ is positive.

Theorem 2. *Assume that Ω is bounded. Under the same assumptions for $\vec{A}$ and V as in theorem 1, the Fourier series (4) converges absolutely and uniformly on Ω for each function f in the domain of the operator $(H_m)_F^\sigma$ for $\sigma > \frac{n}{4}$.*

Some survey of the results concerning the convergence of the spectral resolutions associated with the elliptic differential operators can be found in [1–7, 12–15] and references therein. We mention only the papers that are of interest from the viewpoint of this paper.

In the present work, we follow partly the techniques appearing in [2, 3, 12, 14]. We use the estimates for the fundamental solution of the Laplace operator and obtain the estimates for the fundamental solution of the magnetic Schrödinger operator. Let us remark here that such estimates for the Schrödinger operator (in the case when the magnetic potential $\vec{A}$ is equal to zero) or for the magnetic Schrödinger operator when the magnetic potential $\vec{A}$ is a constant can be found in [11] (for the Schrödinger operator see also [14]). Due to these estimates, we obtain the estimates for the parametrix and prove the convergence (absolute and uniform) of the Fourier series in terms of eigenfunctions. We use the techniques involving fractional powers of the positive self-adjoint operator and J von Neumann's spectral theorem (see, for example, [14]). It might be mentioned here that the estimates for the fundamental solution of the magnetic Schrödinger operator when the magnetic potential $\vec{A}$ is a function from Sobolev space in consideration have independent interest and, as far as we know, have never appeared in the literature.

2. Fundamental solution and Green's function

Since the function $\vec{A}$ belongs to Sobolev class $\left(W_s^1(\Omega)\right)^n$, we can rewrite the operator (1) in the form

$$H_m u(x) \equiv -\Delta u(x) - 2\mathrm{i}\vec{A}(x)\cdot\nabla u(x) + \tilde{q}(x)u(x), \tag{6}$$

where $\tilde{q}(x)$ denotes the following function:

$$\tilde{q} = |\vec{A}|^2 + V - \mathrm{i}\nabla\cdot\vec{A}.$$

The Sobolev embedding theorem allow us to conclude that $\tilde{q}$ belongs to $L^s(\Omega)$ with the same s as before.

We look for the fundamental solution $F(x, y, \lambda)$ of the operator $H_m + \lambda I$, for λ positive and large enough, as the solution of the integral equation

$$F(x, y, \lambda) = F_0(x, y, \lambda) + \int_\Omega F_0(x, z, \lambda)(2\mathrm{i}\vec{A}(z)\cdot\nabla_z F(z, y, \lambda) - \tilde{q}(z)F(z, y, \lambda))\,\mathrm{d}z, \tag{7}$$

where $F_0(x, y, \lambda)$ is the fundamental solution of the operator $-\Delta + \lambda I$. It is well known that for any positive λ function F_0 has a form

$$F_0(x, y, \lambda) = (2\pi)^{-\frac{n}{2}}\left(\frac{\sqrt{\lambda}}{|x-y|}\right)^{\frac{n-2}{2}} K_{\frac{n-2}{2}}(\sqrt{\lambda}|x-y|),$$

where $K_\nu(t)$ is the Macdonald function of order ν. Using the well-known properties of the Macdonald function K_ν (see, for example, [17]), we can assert that there is a constant $C_0 > 0$ such that for any $\lambda > 0$ the following inequalities hold:

$$|F_0(x, y, \lambda)| \leqslant C_0|x-y|^{2-n}\,\mathrm{e}^{-|x-y|\sqrt{\lambda}} \tag{8}$$

and

$$|\nabla_x F_0(x, y, \lambda)| \leqslant C_0|x-y|^{1-n}\,\mathrm{e}^{-|x-y|\sqrt{\lambda}}, \tag{9}$$

where $x, y \in R^n$.

Proof of theorem 1. We solve the integral equation (7) by iterations. For $j = 1, 2, \ldots,$ we denote

$$F_j(x, y, \lambda) = \int_\Omega F_0(x, z, \lambda)(2\mathrm{i}\vec{A}(z)\cdot\nabla_z F_{j-1}(z, y, \lambda) - \tilde{q}(z)F_{j-1}(z, y, \lambda))\,\mathrm{d}z. \tag{10}$$

We will prove by induction that there is $\lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0$ and for each $j = 0, 1, 2, \ldots$

$$|F_j(x, y, \lambda)| \leqslant \frac{C_0}{2^j}|x-y|^{2-n}\,\mathrm{e}^{-\frac{1}{2}|x-y|\sqrt{\lambda}} \tag{11}$$

and

$$|\nabla_x F_j(x, y, \lambda)| \leqslant \frac{C_0}{2^j}|x-y|^{1-n}\,\mathrm{e}^{-\frac{1}{2}|x-y|\sqrt{\lambda}}, \tag{12}$$

where $x, y \in \Omega$ and C_0 is as in (8) and (9).

It is clear that (11) and (12) hold for $j = 0$. In order to prove these estimates for any $j \geqslant 1$ consider two cases: $|x-z| \geqslant |z-y|$ and $|x-z| \leqslant |z-y|$. Using the assumption of induction we can obtain uniformly with respect to $x, y \in \Omega$ the estimates

$$\begin{aligned}
|F_{j+1}(x, y, \lambda)| \leqslant{}& \frac{C_0^2}{2^j}\left(\frac{|x-y|}{2}\right)^{2-n}\mathrm{e}^{-\frac{1}{2}|x-y|\sqrt{\lambda}}\int_{|x-z|\geqslant|z-y|}(2|\vec{A}(z)||z-y|^{1-n}\\
&+|\tilde{q}(z)||z-y|^{2-n})\,\mathrm{e}^{-\frac{1}{2}|z-y|\sqrt{\lambda}}\,\mathrm{d}z + \frac{C_0^2}{2^j}\left(\frac{|x-y|}{2}\right)^{2-n}\mathrm{e}^{-\frac{1}{2}|x-y|\sqrt{\lambda}}\\
&\times\int_{|x-z|\leqslant|z-y|}\mathrm{e}^{-\frac{1}{2}|x-z|\sqrt{\lambda}}(2|\vec{A}(z)||x-z|^{1-n}+|\tilde{q}(z)||x-z|^{2-n})\,\mathrm{d}z
\end{aligned} \tag{13}$$

and (using the same techniques as above)

$$|\nabla_x F_{j+1}(x,y,\lambda)| \leqslant \frac{C_0^2}{2^j}\left(\frac{|x-y|}{2}\right)^{1-n} \mathrm{e}^{-\frac{1}{2}|x-y|\sqrt{\lambda}} \int_{|x-z|\geqslant|z-y|} (2|\vec{A}(z)||z-y|^{1-n} + |\tilde{q}(z)||z-y|^{2-n})\,\mathrm{e}^{-\frac{1}{2}|z-y|\sqrt{\lambda}}\,\mathrm{d}z + \frac{C_0^2}{2^j}\left(\frac{|x-y|}{2}\right)^{1-n}\mathrm{e}^{-\frac{1}{2}|x-y|\sqrt{\lambda}} \times \int_{|x-z|\leqslant|z-y|} \mathrm{e}^{-\frac{1}{2}|x-z|\sqrt{\lambda}}(2|\vec{A}(z)||x-z|^{1-n} + |\tilde{q}(z)||x-z|^{2-n})\,\mathrm{d}z. \tag{14}$$

The estimates (13) and (14) show that it suffices to prove that there is $\lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0$ and $x, y \in \Omega$ the following inequality holds:

$$C_0 \int_{|x-z|\geqslant|z-y|} (2|\vec{A}(z)||z-y|^{1-n} + |\tilde{q}(z)||z-y|^{2-n})\,\mathrm{e}^{-\frac{1}{2}|z-y|\sqrt{\lambda}}\,\mathrm{d}z + C_0 \int_{|x-z|\leqslant|z-y|} (2|\vec{A}(z)||x-z|^{1-n} + |\tilde{q}(z)||x-z|^{2-n})\,\mathrm{e}^{-\frac{1}{2}|x-z|\sqrt{\lambda}}\,\mathrm{d}z \leqslant \frac{1}{2^n}. \tag{15}$$

The assumptions for $\vec{A}$ and V and Hölder's inequality allow us to obtain the following estimates (which are uniform in y and x from Ω, respectively):

$$\int |\vec{A}(z)||z-\cdot|^{1-n}\,\mathrm{e}^{-\frac{1}{2}|z-\cdot|\sqrt{\lambda}}\,\mathrm{d}z \leqslant \left(\int |\vec{A}(z)|^p\,\mathrm{d}z\right)^{\frac{1}{p}} \times \left(\int |z-\cdot|^{p'(1-n)}\,\mathrm{e}^{-\frac{p'}{2}|z-\cdot|\sqrt{\lambda}}\,\mathrm{d}z\right)^{\frac{1}{p'}} \leqslant C(\sqrt{\lambda})^{\frac{n}{p}-1}, \tag{16}$$

where p can be chosen such that $p > n$, and

$$\int |\tilde{q}(z)||z-\cdot|^{2-n}\,\mathrm{e}^{-\frac{1}{2}|z-\cdot|\sqrt{\lambda}}\,\mathrm{d}z \leqslant \left(\int |\tilde{q}(z)|^{p_1}\,\mathrm{d}z\right)^{\frac{1}{p_1}} \times \left(\int |z-\cdot|^{p_1'(2-n)}\,\mathrm{e}^{-\frac{p_1'}{2}|z-\cdot|\sqrt{\lambda}}\,\mathrm{d}z\right)^{\frac{1}{p_1'}} \leqslant C(\sqrt{\lambda})^{\frac{n}{p_1}-2}, \tag{17}$$

where p_1 can be chosen such that $p_1 > \frac{n}{2}$, and the constant C in (16) and (17) does not depend on x and y. Now (15) follows immediately from (16) and (17). Thus, the estimates (11) and (12) are completely proved by induction.

Since the solution $F(x,y,\lambda)$ of the integral equation (7) is given by the series

$$F(x,y,\lambda) = \sum_{j=0}^{\infty} F_j(x,y,\lambda),$$

where $F_j(x,y,\lambda)$ are defined by (10), then the estimates (11) and (12) prove also theorem 1. □

Remark 1. It is easy to check that the constant C in theorem 1 is equal to $2C_0$, where C_0 is as in (8) and (9).

As a consequence of theorem 1, we can obtain the estimates for the derivatives of order 1 of the fundamental solution $F(x,y,\lambda)$. The following corollary holds.

Corollary 1. *Under the same assumptions for $\vec{A}$ and V as in theorem 1, there exists a constant $C > 0$ such that for all $x, y \in \Omega$*

$$|\nabla_x F(x,y,\lambda)| \leqslant C|x-y|^{1-n}\,\mathrm{e}^{-\frac{1}{2}|x-y|\sqrt{\lambda}} \tag{18}$$

with $\lambda \geqslant \lambda_0$ (where λ_0 is as in the theorem).

Now we are in a position to introduce the Green's function of the operator $(H_m)_F + \lambda I$. If λ is sufficiently large then the operator $(H_m)_F + \lambda I$ is positive and its inverse

$$((H_m)_F + \lambda I)^{-1} : L^2(\Omega) \to L^2(\Omega) \tag{19}$$

is a bounded operator. It is an integral operator with kernel denoted by $G(x, y, \lambda)$. If we use for this integral operator the symbol $\widehat{G}(\lambda)$ then we have

$$((H_m)_F + \lambda I)\widehat{G}(\lambda) = I, \qquad \widehat{G}(\lambda)((H_m)_F + \lambda I) = I, \qquad G(x, y, \lambda) = \overline{G(y, x, \lambda)}. \tag{20}$$

Definition 1. *The kernel $G(x, y, \lambda)$ of the integral operator $\widehat{G}(\lambda)$ is called the Green's function of the operator $(H_m)_F + \lambda I$.*

For $\tau > 0$, let Ω^τ and $\Omega^{\frac{\tau}{2}}$ be compact sets, each of them having a smooth boundary, with $\Omega^\tau \subset \Omega^{\frac{\tau}{2}} \subset \Omega$ such that

$$d(\Omega^\tau, \partial\Omega) \geqslant \tau, \qquad d(\Omega^{\frac{\tau}{2}}, \partial\Omega) \geqslant \frac{\tau}{2}$$

and

$$d(\Omega^\tau, \partial\Omega^{\frac{\tau}{2}}) = \frac{\tau}{2}.$$

Here $d(X, Y)$ denotes the distance between the sets X and Y.

Let $F(x, y, \lambda)$ be a fundamental solution of the operator $(H_m)_F + \lambda I$ for $x, y \in \Omega$ and λ sufficiently large. We choose the function $\chi \in C_0^\infty(\Omega)$ such that

$$\chi(x) = \begin{cases} 1, & x \in \Omega^\tau \\ 0, & x \in \Omega \backslash \Omega^{\frac{\tau}{2}} \end{cases}$$

and set

$$E(x, y, \lambda) = \chi(x) F(x, y, \lambda). \tag{21}$$

By this equation, the function $E(x, y, \lambda)$ is well defined for all $x, y \in \Omega$. Clearly, $E(x, y, \lambda) = F(x, y, \lambda)$ for $x \in \Omega^\tau$, $y \in \Omega$. We will show that $E(x, y, \lambda)$ is a parametrix for $(H_m)_F + \lambda I$. To prove this, let us introduce the function

$$Q(x, y, \lambda) := G(x, y, \lambda) - E(x, y, \lambda) \tag{22}$$

and the corresponding integral operator with kernel $Q(x, y, \lambda)$

$$\widehat{Q}(\lambda) := \widehat{G}(\lambda) - \widehat{E}(\lambda), \tag{23}$$

where $\widehat{E}(\lambda)$ and $\widehat{G}(\lambda)$ are integral operators in $L^2(\Omega)$ with kernels $E(x, y, \lambda)$ and $G(x, y, \lambda)$, respectively. Then it follows from (22) that

$$((H_m)_F + \lambda I)\widehat{E}(\lambda) = I + \widehat{P}_1(\lambda), \tag{24}$$

where

$$\widehat{P}_1(\lambda) = -((H_m)_F + \lambda I)\widehat{Q}(\lambda)$$

and

$$\widehat{Q}(\lambda) = -\widehat{G}(\lambda)\widehat{P}_1(\lambda). \tag{25}$$

If we denote by $P_1(x, y, \lambda)$ the kernel of the integral operator $\widehat{P}_1(\lambda)$, then for any $f \in L^2(\Omega)$,

$$\widehat{Q}(\lambda) f(x) = -\int_\Omega \left(\int_\Omega G(x, z, \lambda) P_1(z, y, \lambda)\, \mathrm{d}z \right) f(y)\, \mathrm{d}y \tag{26}$$

and the kernel $Q(x, y, \lambda)$ (see (22)) has the form

$$Q(x, y, \lambda) = -\int_\Omega G(x, z, \lambda) P_1(z, y, \lambda)\, dz, \tag{27}$$

where $x, y \in \Omega$. As a matter of fact, we cannot characterize and estimate the kernel $P_1(x, y, \lambda)$ from (24)–(25). That is why we will proceed a little bit differently, as follows. The equality (21) implies that in the sense of distributions the following representation holds:

$$(H_m(x, D) + \lambda I) E(x, y, \lambda) = \chi(x)\delta(x - y) + P(x, y, \lambda), \tag{28}$$

where $x \in \Omega$ ($y \in \Omega$ is considered here as a parameter) and λ is sufficiently large. The function $P(x, y, \lambda)$ in (28) will be of the form

$$P(x, y, \lambda) = \sum_{0<|\alpha|\leqslant 2} \frac{D^\alpha \chi(x)}{\alpha!} H_m^{(\alpha)}(x, D) F(x, y, \lambda) \tag{29}$$

with the differential operator $H_m^{(\alpha)}(x, D)$ having the symbol $H_m^{(\alpha)}(x, -i\xi) = \partial_\xi^\alpha H_m(x, -i\xi)$. It is the polynomial in $\xi \in R^n$ of order $\leqslant 1$ and therefore the differential operators $H_m^{(\alpha)}(x, D)$ are of order $\leqslant 1$. The concrete form of the differential operators $H_m^{(\alpha)}(x, D)$ allows us to estimate the function $P(x, y, \lambda)$ (in comparison with P_1). Indeed, the representation (29) and corollary 1 imply that the following estimate holds:

$$|P(x, y, \lambda)| \leqslant C(|x - y|^{1-n} + |\vec{A}(x)||x - y|^{2-n})\, e^{-\frac{1}{2}|x-y|\sqrt{\lambda}} \tag{30}$$

for all $x, y \in \Omega$ and for all $\lambda \geqslant \lambda_0$.

Now we need the following lemma.

Lemma 1. *For all* $x, y \in \Omega$

$$\chi(y) G(x, y, \lambda) = \chi(x) F(x, y, \lambda) - \int_\Omega G(x, u, \lambda) P(u, y, \lambda)\, du, \tag{31}$$

where P *is as in (29) and* χ *is defined as in (21).*

Proof. We can rewrite (28) in the operator form as

$$((H_m)_F + \lambda I)\widehat{E}(\lambda) = \chi I + \widehat{P}(\lambda)$$

or (using (22))

$$\widehat{P}(\lambda) = (1 - \chi) I - ((H_m)_F + \lambda I)\widehat{Q}(\lambda).$$

The latter equation implies

$$\widehat{Q}(\lambda) = \widehat{G}(\lambda)((1 - \chi)I) - \widehat{G}(\lambda)\widehat{P}(\lambda)$$

and therefore (using (22) again)

$$\widehat{G}(\lambda)(\chi I) = \widehat{E}(\lambda) - \widehat{G}(\lambda)\widehat{P}(\lambda).$$

But this is equivalent to (31). Thus, this lemma is proved. □

In order to obtain the needed estimates of the Green's function, let us introduce new functions $\widetilde{F}$ and $\widetilde{G}$ which are obtained from F and G multiplying by

$$e^{\frac{1}{4}|x-y|\sqrt{\lambda}} |x - y|^{n-2}.$$

Then equation (31) and estimate (30) formally yield the following estimate:

$$\sup_{x,y\in\Omega} |\chi(y)\widetilde{G}(x, y, \lambda)| \leqslant \sup_{x,y\in\Omega} |\widetilde{F}(x, y, \lambda)| + \sup_{x,y\in\Omega} |\widetilde{G}(x, y, \lambda)| \times \sup_{x,y\in\Omega}\left(|x - y|^{n-2}\int_\Omega |x - z|^{2-n}(|z - y|^{1-n} + |\vec{A}(z)||z - y|^{2-n})\, e^{-\frac{1}{4}|z-y|\sqrt{\lambda}}\, dz\right). \tag{32}$$

Considering two possibilities $|x-z| \leqslant |z-y|$ and $|x-z| \geqslant |z-y|$ the value in the latter brackets can be estimated from above by

$$C\int_{|x-z|\geqslant|z-y|}(|z-y|^{1-n}+|\vec{A}(z)||z-y|^{2-n})\,\mathrm{e}^{-\frac{1}{4}|z-y|\sqrt{\lambda}}\,\mathrm{d}z$$
$$+C\int_{|x-z|\leqslant|z-y|}(|x-z|^{1-n}+|\vec{A}(z)||x-z|^{2-n})\,\mathrm{e}^{-\frac{1}{4}|x-z|\sqrt{\lambda}}\,\mathrm{d}z \leqslant \frac{C}{\sqrt{\lambda}},$$

where the constant C does not depend on x and y. This estimate allows us to get from (32) that

$$\sup_{x,y\in\Omega}|\widetilde{G}(x,y,\lambda)| \leqslant \sup_{x,y\in\Omega}|\widetilde{F}(x,y,\lambda)| + \frac{C}{\sqrt{\lambda}}\sup_{x,y\in\Omega}|\widetilde{G}(x,y,\lambda)|. \tag{33}$$

Since

$$\sup_{x,y\in\Omega}|\widetilde{F}(x,y,\lambda)| < \infty,$$

then for λ large enough (33) yields

$$\sup_{x,y\in\Omega}|\widetilde{G}(x,y,\lambda)| < \infty.$$

Thus we proved the following theorem.

Theorem 3. *Assume that* $\vec{A}(x) \in \left(W_s^1(\Omega)\right)^n$ *and* $V(x) \in L^s(\Omega)$ *for some* $\frac{n}{2} < s \leqslant \infty$. *Then there exists* $\lambda_0 > 0$ *such that for any* $\lambda \geqslant \lambda_0$, *the Green's function* $G(x,y,\lambda)$ *of the operator* $H_m + \lambda I$ *satisfies the following estimate:*

$$|G(x,y,\lambda)| \leqslant C|x-y|^{2-n}\,\mathrm{e}^{-\frac{1}{4}|x-y|\sqrt{\lambda}}, \tag{34}$$

where a positive constant C *does not depend on* $x, y \in \Omega$ *and* λ.

3. Convergence of Fourier series

Without loss of generality, we assume in this chapter that $(H_m)_F$ is positive. We assume also in this chapter that Ω is bounded. Then by the J von Neumann's spectral theorem for $(H_m)_F + \mu I$, where $\mu \geqslant \lambda_0$ with λ_0 is as in theorem 3, the following representation holds:

$$((H_m)_F + \mu I)^s f(x) = \int_0^\infty (\lambda+\mu)^s\,\mathrm{d}E_\lambda f(x), \tag{35}$$

where s is real and $\{E_\lambda\}_{\lambda=0}^\infty$ is the spectral resolution corresponding to $(H_m)_F$. The domain of the operator (35) can be described as

$$D((H_m)_F^s) = \left\{ f \in L^2(\Omega) : \int_0^\infty \lambda^{2s}\,\mathrm{d}(E_\lambda f, f) < \infty \right\}. \tag{36}$$

In the case of discrete spectrum, the spectral projector E_λ has the form

$$E_\lambda f(x) = \sum_{\lambda_k<\lambda} f_k\varphi_k(x), \tag{37}$$

where f_k are the Fourier coefficients of f with respect to the orthonormal basis $\{\varphi_k(x)\}_{k=1}^\infty$ of eigenfunctions of $(H_m)_F$. Hence relations (35) and (36) become

$$(H_m)_F^s f(x) = \sum_{k=1}^\infty \lambda_k^s f_k\varphi_k(x), \tag{38}$$

$$D((H_m)_F^s) = \left\{ f \in L^2(\Omega) : \sum_{k=1}^{\infty} |f_k|^2 \lambda_k^{2s} < \infty \right\}. \tag{39}$$

In addition, we need a special representation for the negative fractional powers of $(H_m)_F$. Let us start with the spectral representation for the Green operator $\widehat{G}(t)$ (see (20)):

$$\widehat{G}(t) f(x) = \int_0^\infty \frac{1}{\lambda + t} \, \mathrm{d}E_\lambda f(x), \qquad t > 0.$$

If we assume that $0 < \tau < 1$ then using the well-known properties of Euler beta-function and (35), one can obtain

$$\begin{aligned} ((H_m)_F + \mu I)^{-\tau} f(x) &= \int_0^\infty (\lambda + \mu)^{-\tau} \, \mathrm{d}E_\lambda f(x) \\ &= \frac{\sin(\tau\pi)}{\pi} \int_0^\infty \left(\int_0^\infty \frac{t^{-\tau}}{\lambda + \mu + t} \, \mathrm{d}t \right) \mathrm{d}E_\lambda f(x) \\ &= \frac{\sin(\tau\pi)}{\pi} \int_0^\infty t^{-\tau} \left(\int_0^\infty \frac{1}{\lambda + \mu + t} \, \mathrm{d}E_\lambda f(x) \right) \mathrm{d}t \\ &= \frac{\sin(\tau\pi)}{\pi} \int_0^\infty t^{-\tau} \hat{G}(\mu + t) f(x) \, \mathrm{d}t. \end{aligned} \tag{40}$$

Therefore, it can be concluded that $((H_m)_F + \mu I)^{-\tau}, 0 < \tau < 1$, is an integral operator. Let us denote the kernel of this operator by $K_\tau(x, y)$. The following lemma holds.

Lemma 2. *If* $0 < \tau < 1$ *then the kernel* $K_\tau(x, y)$ *satisfies the estimate*

$$|K_\tau(x, y)| \leqslant C|x - y|^{2\tau - n} \, \mathrm{e}^{-\frac{1}{8}|x-y|\sqrt{\mu}}, \tag{41}$$

where $x, y \in \Omega$ *and* C *is a positive constant which does not depend on* x *and* y.

Proof. Using (40) we can get

$$K_\tau(x, y) = \frac{\sin(\tau\pi)}{\pi} \int_0^\infty t^{-\tau} G(x, y, \mu + t) \, \mathrm{d}t.$$

The inequality (34) then implies that

$$\begin{aligned} |K_\tau(x, y)| &\leqslant C|x - y|^{2-n} \int_0^\infty t^{-\tau} \, \mathrm{e}^{-\frac{1}{4}|x-y|\sqrt{\mu+t}} \, \mathrm{d}t \\ &\leqslant C|x - y|^{2-n} \, \mathrm{e}^{-\frac{1}{8}|x-y|\sqrt{\mu}} \int_0^\infty t^{-\tau} \, \mathrm{e}^{-\frac{1}{8}|x-y|\sqrt{t}} \, \mathrm{d}t. \end{aligned}$$

Changing the variable of integration in the latter integral we can easily obtain (41). Thus, lemma 2 is proved. □

Lemma 3. *Assume that* $0 < \tau_1, \tau_2 < 1$. *Then the operator*

$$((H_m)_F + \mu I)^{-(\tau_1 + \tau_2)} := ((H_m)_F + \mu I)^{-\tau_1} \cdot ((H_m)_F + \mu I)^{-\tau_2}$$

is an integral operator with kernel $K_{\tau_1+\tau_2}(x, y)$ *which satisfies the estimate*

$$\left| K_{\tau_1+\tau_2}(x, y) \right| \leqslant C|x - y|^{2(\tau_1+\tau_2)-n} \, \mathrm{e}^{-\frac{1}{8}|x-y|\sqrt{\mu}}. \tag{42}$$

where $x, y \in \Omega$.

Proof. It is clear that the considered operator is an integral operator and its kernel $K_{\tau_1+\tau_2}(x, y)$ can be estimated as (see (41))

$$\begin{aligned}\left|K_{\tau_1+\tau_2}(x, y)\right| &\leqslant \int_\Omega \left|K_{\tau_1}(x, z)\right|\left|K_{\tau_2}(z, y)\right| \mathrm{d}z \\ &\leqslant C\,\mathrm{e}^{-\frac{1}{8}|x-y|\sqrt{\mu}} \int_\Omega |x-z|^{2\tau_1-n}|z-y|^{2\tau_2-n}\,\mathrm{d}z,\end{aligned} \tag{43}$$

where $x, y \in \Omega$. Using now the well-known estimates for convolution in (43) of the weak singularities we can obtain (42). Here we have used the fact that Ω is bounded. Hence, lemma 3 is proved. □

We are now in a position to formulate the main results of this paragraph.

Lemma 4 (main lemma). *Assume that* $\vec{A}(x) \in \left(W_s^1(\Omega)\right)^n$, $V(x) \in L^s(\Omega)$ *for some* $\frac{n}{2} < s \leqslant \infty$ *and* $\sigma > n/4, n \geqslant 2$. *Then for any function* $f(x) \in L^2(\Omega)$ *the following inequality holds:*

$$\|((H_m)_F + \mu I)^{-\sigma} f\|_{L^\infty(\Omega)} \leqslant C\mu^{\frac{n}{4}-\sigma}\|f\|_{L^2(\Omega)}, \tag{44}$$

where $\mu \geqslant \lambda_0$ *with* λ_0 *as in theorem 3.*

Proof. Let us recall that the operator $((H_m)_F + \mu I)^{-\sigma}$ for any positive σ can be represented by the following composition:

$$((H_m)_F + \mu I)^{-\sigma} = ((H_m)_F + \mu I)^{-\tau_1} \cdots ((H_m)_F + \mu I)^{-\tau_m},$$

where $\tau_1 + \cdots + \tau_m = \sigma$ and $0 < \tau_j < 1$ for each $j = 1, 2, \ldots, m$. Hence, this operator is the integral operator with kernel K_σ, and lemma 3 implies that this kernel satisfies the estimates (42) with the substitution σ instead of $\tau_1+\tau_2$. Then applying (42) and the Cauchy–Bunjakovskii inequality we may obtain

$$\begin{aligned}|((H_m)_F + \mu I)^{-\sigma} f(x)| &\leqslant C\left(\int_\Omega |x-y|^{4\sigma-2n}\,\mathrm{e}^{-\frac{1}{8}|x-y|\sqrt{\mu}}\,\mathrm{d}y\right)^{\frac{1}{2}} \|f\|_{L^2(\Omega)} \\ &\leqslant C\mu^{\frac{n}{4}-\sigma}\|f\|_{L^2(\Omega)}.\end{aligned}$$

This completes the proof of lemma 4. □

Corollary 2. *Assume that* $\sigma > n/4$. *There is a constant* $C > 0$ *depending only on* Ω*, such that the estimate*

$$\sum_{k=1}^{\infty} \frac{|\varphi_k(x)|^2}{(\lambda_k + \mu)^{2\sigma}} \leqslant C\mu^{\frac{n}{2}-2\sigma} \tag{45}$$

holds uniformly in $x \in \Omega$ *and* $\mu \geqslant \lambda_0$.

Proof. By the spectral theorem and relation (38), we can rewrite the inequality (44) in the form

$$|((H_m)_F + \mu I)^{-\sigma} f(x)| = \left|\sum_{k=1}^{\infty}(\lambda_k + \mu)^{-\sigma} f_k\varphi_k(x)\right| \leqslant C\mu^{\frac{n}{4}-\sigma}\left(\sum_{k=1}^{\infty}|f_k|^2\right)^{1/2},$$

where f_k are the Fourier coefficients of f with respect to the orthonormal basis $\{\varphi_k(x)\}_{k=1}^{\infty}$. Hence, by duality in the Hilbert space l^2 we may conclude that (45) holds. It proves this lemma. □

Remark 2. The inequality (45) has an independent interest since it gives the 'bundle' estimate of the eigenfunctions in the form

$$\sum_{\lambda\leqslant\lambda_k<2\lambda} |\varphi_k(x)|^2 \leqslant C\lambda^{\frac{n}{2}} \tag{46}$$

uniformly in $x \in \Omega$ and $\lambda \geqslant \lambda_0$.

We are now ready to prove theorem 2.

Proof of theorem 2. Suppose that $\sigma > n/4$. The Cauchy–Bunjakovskii inequality and lemma 4 lead to the inequality

$$\sum_{k=1}^{\infty} |f_k\varphi_k(x)| \leqslant \left(\sum_{k=1}^{\infty}(\lambda_k+\mu)^{-2\sigma}|\varphi_k(x)|^2\right)^{1/2}\left(\sum_{k=1}^{\infty}|f_k|^2(\lambda_k+\mu)^{2\sigma}\right)^{1/2}$$
$$\leqslant C\mu^{\frac{n}{4}-\sigma}\left(\sum_{k=1}^{\infty}|f_k|^2(\lambda_k+\mu)^{2\sigma}\right)^{1/2} \leqslant C\left(\sum_{k=1}^{\infty}|f_k|^2\lambda_k^{2\sigma}\right)^{1/2}$$

for any fixed $\mu \geqslant \lambda_0$. But the latter series converges since $f \in D\big((H_m)_F^{\sigma}\big)$ (see (39)). Thus, theorem 2 is proved. □

Let us assume now that $\vec{A}(x)$ and $V(x)$ satisfy the conditions

$$\begin{cases} \vec{A}(x) \in \left(W_2^1(\Omega)\right)^3, & V(x) \in L^2(\Omega), & & n = 3, \\ \vec{A}(x) \in \left(W_s^1(\Omega)\right)^n, & V(x) \in L^s(\Omega), & \dfrac{n}{2} < s \leqslant \infty, & n \geqslant 4. \end{cases} \tag{47}$$

The following result is valid.

Theorem 4. *Suppose that $\vec{A}(x)$ and $V(x)$ satisfy the conditions (47). Then for each function $f \in \overset{\circ}{W}{}_2^2(\Omega)$*

$$\lim_{\lambda\to+\infty} \|f - E_\lambda f\|_{W_2^2} = 0, \tag{48}$$

where E_λ is the spectral resolution corresponding to $(H_m)_F$.

Proof. Using the Sobolev embedding theorem we easily conclude that the conditions (47) imply the inclusion

$$\overset{\circ}{W}{}_2^2(\Omega) \subset D((H_m)_F).$$

And for any $f \in \overset{\circ}{W}{}_2^2(\Omega)$, the following inequality holds:

$$\|(H_m)_F f\|_{L^2(\Omega)} \leqslant C\|f\|_{W_2^2(\Omega)}.$$

Moreover, we may assert that the operator $(H_m)_F + \mu I$ is invertible for μ large enough. Indeed, since the function

$$h(x) := ((H_m)_F + \mu I)f(x), \qquad f(x) \in \overset{\circ}{W}{}_2^2(\Omega)$$

belongs to $L^2(\Omega)$, we have the representation for $f(x)$

$$f(x) = -(-\Delta+\mu I)_F^{-1}h(x) + 2\mathrm{i}(-\Delta+\mu I)_F^{-1}(\vec{A}\cdot\nabla f)(x) - (-\Delta+\mu I)_F^{-1}(\tilde{q}f)(x),$$

where $(-\Delta+\mu I)_F$ denotes the Friedrichs self-adjoint extension for $-\Delta+\mu I$ in $L^2(\Omega)$. Using again the Sobolev embedding theorem and the conditions (47) we may conclude that the functions h, $\vec{A}\cdot\nabla f$ and $\tilde{q}f$ belong to $L^2(\Omega)$. The results of [1] (see also [16]) yield that the operator $(-\Delta + \mu I)_F^{-1}$ exists with small operator norm for μ large enough. This fact and the latter identity imply that for μ large enough the operator $((H_m)_F + \mu I)$ is invertible and for any $h \in L^2(\Omega)$ the following inequality holds:

$$\|((H_m)_F + \mu I)^{-1} h\|_{W_2^2(\Omega)} \leqslant C\|h\|_{L^2(\Omega)}. \tag{49}$$

Using now (49) for $f \in \overset{\circ}{W}{}_2^2(\Omega)$ we have

$$\begin{aligned}\|f - E_\lambda f\|_{W_2^2(\Omega)} &= \|((H_m)_F + \mu I)^{-1}(h - E_\lambda h)\|_{W_2^2(\Omega)} \\ &\leqslant C\|h - E_\lambda h\|_{L^2(\Omega)} \to 0, \quad \lambda \to +\infty,\end{aligned}$$

where $h := ((H_m)_F + \mu I) f \in L^2(\Omega)$ and the convergence to zero in the last term follows from the J von Neumann spectral theorem. Hence, theorem 4 is proved. □

Corollary 3. *Let $n = 3$. Then for any $f \in \overset{\circ}{W}{}_2^\alpha(\Omega)$ with $\alpha > \frac{3}{2}$ the Fourier series (4) converges absolutely and uniformly on Ω.*

Let us conclude this paper by two important remarks.

Remark 3. There are the analogs of theorems 1 and 2 for the two-dimensional case.

Theorem 5. *Suppose that $\vec{A}(x) \in \left(W_s^1(\Omega)\right)^2$ and $V(x) \in L^s(\Omega)$ for some $1 < s \leqslant \infty$. Then there exist $C > 0$ and $\lambda_0 > 0$ such that for any $\lambda \geqslant \lambda_0$, the operator $H_m + \lambda I$ has a fundamental solution $F(x, y, \lambda)$ which satisfies the following estimate:*

$$|F(x, y, \lambda)| \leqslant C(1 + |\log(|x - y|\sqrt{\lambda})|)\, \mathrm{e}^{-\frac{1}{2}|x-y|\sqrt{\lambda}},$$

for all $x, y \in \Omega$.

Theorem 6. *Assume that Ω is bounded. Under the same assumptions for $\vec{A}(x)$ and $V(x)$ as in theorem 5, the Fourier series (4) converges absolutely and uniformly on Ω for each function $f(x)$ in the domain of the operator $(H_m)_F^\sigma$ for $\sigma > \frac{1}{2}$.*

Remark 4. It is not difficult to check that theorem 4 is also valid for the two-dimensional case if we assume that $\vec{A}(x) \in \left(W_2^1(\Omega)\right)^2$ and $V(x) \in L^2(\Omega)$. Moreover, under some additional conditions for $\vec{A}(x)$ and $V(x)$, the following theorem holds (see [13, 16] for details) for any dimension $n \geqslant 2$.

Theorem 7. *Assume that $\vec{A}(x) \in \left(W_2^{2l+1}(\Omega)\right)^n$ and $V(x) \in W_2^{2l}(\Omega)$, where $n \geqslant 2$ and $l = \left[\frac{n}{4}\right]$ is an entire part of $\frac{n}{4}$. Then the Fourier series (4) converges absolutely and uniformly on Ω for each function $f(x)$ from Sobolev space $\overset{\circ}{W}{}_2^\alpha(\Omega)$ for $\alpha > \frac{n}{2}$.*

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